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Minimal Extensions in Tensor Product Spaces

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Let X, Y be two separable Banach spaces and let $V \subset X$ and $W \subset Y$ be finite dimensional subspaces. Suppose that $V \subset S \subset X$, $W \subset Z \subset Y$ and let $M \in \mathscr{L}(S, V)$, $N \in \mathscr{L}(Z, W)$. We will prove that if α is a reasonable, uniform crossnorm on $X \otimes Y$ then

$$\lambda_{M \otimes N}(V \otimes_\alpha W, X \otimes_\alpha Y) = \lambda_M(V, X)\, \lambda_N(W, Y).$$

Here for any Banach space X, $V \subset S \subset X$ and $M \in \mathscr{L}(S, V)$

$$\lambda_M(V, X) = \inf\{\|P\| : P \in \mathscr{L}(X, V), P|_S = M\}.$$

Also some applications of the above mentioned result will be presented.

1

Let X be a Banach space and let $V \subset X$ be a linear subspace. An operator $P \in \mathscr{L}(X, V)$ is called a projection if $P|_V = \mathrm{id}_V$. The set of all projections from X onto V will be denoted by $\mathscr{P}(X, V)$.

A projection $P_o \in \mathscr{P}(X, V)$ is called *minimal* if

$$\|P_o\| = \lambda(V, X) = \inf\{\|P\| : P \in \mathscr{P}(X, V)\}. \tag{1.1}$$

The problem of finding formulas for minimal projections is related to the Hahn–Banach Theorem, as well as to the problem of producing a “good” linear replacement of an $x \in X$ by a certain element from V, because of the inequality

$$\|x - Px\| \leqslant \|\mathrm{Id} - P\| \operatorname{dist}(x, V) \leqslant (1 + \|P\|) \operatorname{dist}(x, V),$$

where $P \in \mathscr{P}(X, V)$. For more information about minimal projections the reader is referred to references included in this paper.

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An analogous problem can be posed in the case of a fixed action $M \in \mathscr{L}(S, V)$ where $V \subset S \subset X$. In this case we want to find an extension of M onto X having the smallest norm, which is clearly the operator version of the Hahn–Banach Theorem. As in the case of projections we denote

$$\mathscr{P}_M(X, V) = \{P \in \mathscr{L}(X, V) : P|_S = M\} \tag{1.2}$$

and

$$\lambda_M(V, X) = \inf\{\|P\| : P \in \mathscr{P}_M(X, V)\}. \tag{1.3}$$

An extension $P \in \mathscr{P}_M(X, V)$ is called *a minimal extension* if

$$\|P\| = \lambda_M(V, X). \tag{1.4}$$

If $S = V$ and $M \in \mathscr{L}(V)$ then by the absolute extension constant we denote a number

$$\lambda_M(V) = \sup\{\lambda_M(V, X) : V \subset X\}. \tag{1.5}$$

If $M = \mathrm{id}_V$, $\lambda_{\mathrm{id}_V}(V, X)$ is called the relative projection constant and $\lambda_{\mathrm{id}_V}(V)$ the absolute projection constant. In the sequel we will write for brevity $\lambda(V, X)$ instead of $\lambda_{\mathrm{id}_V}(V, X)$ and $\lambda(V)$ instead of $\lambda_{\mathrm{id}_V}(V)$.

The aim of this paper is to investigate to the following.

Problem 1.1. Let X, Y be a pair of Banach spaces and let V (W resp.) be a finite-dimensional subspace of X (Y resp.). Suppose that $V \subset S \subset X$ and $W \subset Z \subset Y$. Let $M \in \mathscr{L}(S, V)$ and $N \in \mathscr{L}(Z, W)$ be given. What is the relationship between the constants $\lambda_{M \otimes N}(V \otimes_\alpha W, X \otimes_\alpha Y)$, $\lambda_M(V, X)$, and $\lambda_N(W, Y)$ where α is a reasonable crossnorm on $X \otimes Y$?

We give an answer to this problem in Theorem 2.5. Moreover, in Theorem 2.6 we show that if α is a reasonable, uniform crossnorm then the tensor product of two minimal actions forms a minimal action for $M \otimes N$.

In Section 3 we present some applications of Theorems 2.5 and 2.6, mainly to the case of projections and $X = Y = C[0, 1]$ or $X = Y = L_p[-1, 1]$. We also reprove Theorem 3 from [24] in a simple manner.

Now we introduce some notation and some basic facts which will be of use later.

Definition 1.2. Let X, Y be two Banach spaces and let $x_1, ..., x_m \in X$, $y_1, ..., y_m \in Y$. Then $L = \sum_{i=1}^m x_i \otimes y_i$ can be interpreted as an operator from X^* to Y defined by

$$Lf = \sum_{i=1}^m f(x_i)\, y_i. \tag{1.6}$$

So $X \otimes Y \subset \mathscr{L}(X^*, Y)$ (we put into one equivalence class all expressions of type $\sum_{i=1}^m x_i \otimes y_i$ if they define the same operator).

DEFINITION 1.3. Let α be a norm on $X \otimes Y$. Then $X \otimes_\alpha Y$ means the completion of $X \otimes Y$ with respect to α.

DEFINITION 1.4. Let α be a norm on $X \otimes Y$. α is a *crossnorm* iff

$$\alpha(x \otimes y) = \|x\| \cdot \|y\| \tag{1.7}$$

for $x \in X$, $y \in Y$.

α is *reasonable* if

$$\alpha^*(f \otimes g) := \sup\left\{\sum_{i=1}^m f(x_i)\, g(y_i) : \alpha\left(\sum_{i=1}^m x_i \otimes y_i\right) = 1\right\} = \|f\| \cdot \|g\| \tag{1.8}$$

for any $f \in X^*$ and $g \in Y^*$.

α is *uniform* if for any $A \in \mathscr{L}(X)$, $B \in \mathscr{L}(Y)$,

$$\|A \otimes B\|_\alpha \leqslant \|A\| \cdot \|B\|, \tag{1.9}$$

where $(A \otimes B)(x \otimes y) = Ax \otimes By$ for $x \in X$, $y \in Y$, and

$$\|A \otimes B\|_\alpha := \sup\left\{\alpha\left((A \otimes B)\left(\sum_{i=1}^m x_i \otimes y_i\right)\right) : \alpha\left(\sum_{i=1}^m x_i \otimes y_i\right) = 1\right\}.$$

By $X \otimes_\lambda Y$ we denote the injective tensor product of X and Y, i.e., the completion of $X \otimes Y$ with respect to the norm λ defined by

$$\lambda\left(\sum_{i=1}^n x_i \otimes y_i\right) = \sup\left\{\left\|\sum_{i=1}^n f(x_i)\, y_i\right\| : f \in S^*, \|f\| = 1\right\}. \tag{1.10}$$

Analogously, by $X \otimes_\gamma Y$ we denote the projective tensor product of X and Y. Here the norm γ is given by

$$\gamma(z) = \inf\left\{\sum_{i=1}^n \|x_i\| \cdot \|y_i\| : x_i \in X,\ y_i \in Y,\ z = \sum_{i=1}^n x_i \otimes y_i\right\}. \tag{1.11}$$

Observe that both λ and γ are uniform, reasonable crossnorms (see, e.g., [13, Lemma 1.6, 1.8, and 1.12]). We also need the following

THEOREM 1.5 [13, Corollary 1.14]. *Let S, T be compact, Hausdorff spaces. Then*

$$C(S) \otimes_{\lambda} C(T) = C(S \times T). \tag{1.12}$$

Here for any compact, Hausdorff set T, $C(T)$ denotes the space of all real (or complex) valued functions defined on T equipped with the supremum norm.

THEOREM 1.6 [13, Corollary 1.16]. *If S and T are σ-finite measure spaces, then*

$$L_1(S) \otimes_{\gamma} L_1(T) = L_1(S \times T). \tag{1.13}$$

For a Banach space X, by S_X we denote its unit sphere and by $\operatorname{ext}(S_X)$ the set of extreme points of S_X. We also need

DEFINITION 1.7 (see [7]). $(x^{**}, x^*) \in S_{X^{**}} \times S_{X^*}$ will be called an extremal pair for $Q \in \mathscr{L}(X)$ if

$$(Q^{**}x^{**})(x^*) = \|Q\|, \tag{1.14}$$

where $Q^{**}: X^{**} \to X^{**}$ is the second adjoint extension of Q to X^{**}. The set of all extremal pairs for Q will be denoted by $\mathscr{E}(Q)$.

The main tool in our investigations will be the following

THEOREM 1.8 (see [7, Theorems 1, 2 and Ex. B]). *Let V be a finite-dimensional subspace of a Banach space X (we consider real and complex cases). Let S be a linear subspace of X with $V \subset S \subset X$. Then $P \in \mathscr{P}_M(X, V)$ is a minimal extension if and only if there exists a positive, total mass one, Boreal measure μ supported on $\mathscr{E}(P)$ such that the operator $E_P: X \to X^{**}$ defined by*

$$E_P(z) = \int_{\mathscr{E}(P)} x^*(z)\, x^{**}\, d\mu(x^{**}, x^*) \tag{1.15}$$

takes V into S. Here $M \in \mathscr{L}(S, V)$ is a fixed action and the set $\mathscr{E}(P)$ is equipped with the Cartesian product topology induced by the weak* *topologies on X^{**} and X^*.*

2

We start from a well known

LEMMA 2.1. *Let X, Y be finite-dimensional Banach spaces. Then* $(X \otimes Y)^* = X^* \otimes Y^*$.

Proof. Note that any element $\sum_{i=1}^m f_i \otimes g_i$ defines a linear function on $X \otimes Y$ by

$$\left(\sum_{i=1}^m f_i \otimes g_i\right)(x \otimes y) = \sum_{i=1}^m f_i(x)\, g_i(y).$$

Since $\dim(X \otimes Y) = \dim(X) \dim(Y) = \dim(X^* \otimes Y^*)$, the proof is complete.

LEMMA 2.2. *If X_1 is a dense subspace in a Banach space X and Y_1 is a dense subspace in a Banach space Y, then $X_1 \otimes Y_1$ is dense in $X \otimes_\alpha Y$ for any crossnorm α on $X \otimes Y$.*

Proof. Take any $x \in X$, $y \in Y$. Let $x_n \in X_1$, $y_n \in Y_1$ be so chosen that $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$. Then

$$\begin{aligned}\alpha(x \otimes y - x_n \otimes y_n) &\leqslant \alpha(x \otimes (y - y_n)) + \alpha((x - x_n) \otimes y_n) \\ &= \|x\| \cdot \|y - y_n\| + \|y_n\| \cdot \|x - x_n\|.\end{aligned}$$

Since (y_n) is a bounded sequence, $\alpha(x_n \otimes y_n - x \otimes y) \to 0$. The proof is complete.

LEMMA 2.3. *Let α be a reasonable crossnorm on $X \otimes Y$. If V is a linear subspace of X and W is a linear subspace of Y then α is a reasonable crossnorm on $V \otimes W$.*

Proof. It is clear that α is a crossnorm on $V \otimes W$. Now suppose that there exist $f \in V^*$ and $g \in W^*$ such that

$$\alpha^*(f \otimes g) = \sup\{(f \otimes g)\, z : z \in V \otimes W, \alpha(z) = 1\} > \|f\| \cdot \|g\|.$$

Let F (G resp.) be the Hahn–Banach extension of f to X (g to Y resp.). Note that

$$\begin{aligned}\alpha^*(F \otimes G) &= \sup\{(F \otimes G)\, z : z \in X \otimes Y, \alpha(z) = 1\} \\ &\geqslant \sup\{(F \otimes G)\, z : z \in V \otimes W, \alpha(z) = 1\} \\ &> \|f\| \cdot \|g\| = \|F\| \cdot \|G\|,\end{aligned}$$

a contradiction.

LEMMA 2.4. *Let X, Y be finite-dimensional Banach spaces. Let A be a closed subset of $S_X \times S_{X^*}$ and let B be a closed subset of $S_Y \times S_{Y^*}$. Let μ_A (μ_B resp.) denote a finite Borel measure on A (on B resp.). Let $E_{A,B}$ be an operator from $X \otimes Y$ into itself defined by*

$$E_{A,B}(a \otimes b) = \int_{A \times B} (x^* \otimes y^*)(a \otimes b)(x \otimes y)\, d(\mu_A(x, x^*) \otimes \mu_B(y, y^*)).$$

Then

$$E_{A,B} = E_A \otimes E_B,$$

where $E_A(a) = \int_A x^*(a)\, x\, d\mu_A(x, x^*)$ and $E_B(b) = \int_B y^*(b)\, y\, d\mu_B(y, y^*)$.

Proof. Take $a \in X$ and $b \in Y$. To prove that $E_{A,B}(a \otimes b) = E_A(a) \otimes E_B(b)$, it is necessary to show that for any $F \in (X \otimes Y)^*$

$$F(E_{A,B}(a \otimes b)) = F(E_A(a) \otimes E_B(b)). \tag{2.1}$$

Since X and Y are finite dimensional, by Lemma 2.1, it is necessary to prove (2.1) for $F = f \otimes g$, where $f \in X^*$ and $g \in Y^*$. Note that

$$\begin{aligned}
&(f \otimes g)(E_{A,B}(a \otimes b)) \\
&\quad = (f \otimes g)\left(\int_{A \times B} (x^*(a)\, y^*(b))(x \otimes y)\right) d(\mu_A \otimes \mu_B) \\
&\quad = \int_{A \times B} x^*(a)\, y^*(b)\, f(x)\, g(y)\, d(\mu_A \otimes \mu_B) \\
&\quad = (\text{by Fubini's Theorem}) \left(\int_A x^*(a)\, f(x)\, d\mu_A\right)\left(\int_B y^*(b)\, g(y)\, d\mu_B\right) \\
&\quad = f(E_A(a))\, g(E_B(b)) = (f \otimes g)(E_A(a) \otimes E_B(b)),
\end{aligned}$$

as required. The proof is complete.

THEOREM 2.5. *Let X, Y be separable Banach spaces (complex or real). Suppose $V \subset X$ and $W \subset Y$ are finite dimensional linear subspaces. Let $V \subset S$ and $W \subset Z$, where S is a subspace of X and Z is a subspace of Y, and let $M \in \mathscr{L}(S, V)$, $N \in \mathscr{L}(Z, V)$ be given. If α is a reasonable crossnorm on $X \otimes Y$ then*

$$\lambda_{M \otimes N}(V \otimes_\alpha W, X \otimes_\alpha Y) \geqslant \lambda_M(V, X)\, \lambda_N(W, Y). \tag{2.2}$$

Proof. First suppose that X, Y are finite dimensional. Let $P_1 \in \mathcal{P}_M(X, V)$ and $P_2 \in \mathcal{P}_N(Y, W)$ be minimal extensions. Put

$$A = \mathcal{E}(P_1) \qquad \text{and} \qquad B = \mathcal{E}(P_2) \tag{2.3}$$

(see Definition 1.7). By Theorem 1.8, there exists a Borel measure μ_A (μ_B resp.) supported on A (B resp.) of total mass one, such that

$$E_A(V) \subset S \qquad \text{and} \qquad E_B(W) \subset Z, \tag{2.4}$$

where E_A and E_B are the same as in Lemma 2.4. Let us define a linear functional T on $\mathcal{L}(X \otimes Y)$ by

$$T(L) = \int_{A \times B} (x^* \otimes y^*)(L(x \otimes y)) \, d(\mu_A \otimes \mu_B). \tag{2.5}$$

First we show that $\|T\| \leqslant 1$. To do this, take any $L \in \mathcal{L}(X \otimes Y)$. Note that

$$|T(L)| \leqslant \int_{A \times B} |(x^* \otimes y^*)(L(x \otimes y))| \, d(\mu_A \otimes \mu_B)$$

$$\leqslant \int_{A \times B} \alpha^*(x^* \otimes y^*) \, \|L\| \, \alpha(x \otimes y) \, d(\mu_A \otimes \mu_B) \leqslant \|L\|,$$

since α is a reasonable crossnorm and $(\mu_A \otimes \mu_B)(A \times B) = 1$.

Now, let

$$\mathcal{D} = \text{cl}(\text{span}\{ f(\cdot)(z) \colon z \in V \otimes W, f \in (X \otimes Y)^*, f|_{S \otimes Z} = 0 \}). \tag{2.6}$$

We show that $T|_{\mathcal{D}} = 0$. To do this, take $L \in \mathcal{D}$, $L = f(\cdot)(v \otimes w)$. Then

$$\begin{aligned} T(L) &= \int_{A \times B} (x^* \otimes y^*)(L(x \otimes y)) \, d(\mu_A \otimes \mu_B) \\ &= \int_{A \times B} (x^* \otimes y^*)(f(x \otimes y)(v \otimes w)) \, d(\mu_A \otimes \mu_B) \\ &= \int_{A \times B} f((x^* v)(y^* w)(x \otimes y)) \, d(\mu_A \otimes \mu_B) \\ &= f(E_{A,B}(v \otimes w)) \\ &= (\text{by Lemma 2.4}) \; f(E_A(v) \otimes E_B(w)) \\ &= 0 \qquad (\text{by } (2.4)). \end{aligned}$$

Consequently,

$$\begin{aligned}\lambda_{M\otimes N}(V\otimes W, X\otimes Y) &= \operatorname{dist}(P_1\otimes P_2, \mathscr{D}) \geqslant T(P_1\otimes P_2)\\ &= (\text{by } (2.3))\ \|P_1\|\cdot\|P_2\|\\ &= \lambda_M(V, W)\,\lambda_N(W, Y),\end{aligned}$$

as required.

Now suppose that X, Y are separable Banach spaces of infinite dimension. Let $V\subset X$, $W\subset Y$ be finite-dimensional subspaces. Suppose that S is a subspace of X and Z is a subspace of Y such that $V\subset S\subset X$, $W\subset Z\subset Y$. Since X and Y are separable, $X=\operatorname{cl}(\bigcup_{n=1}^{\infty} X_n)$ and $Y=\operatorname{cl}(\bigcup_{n=1}^{\infty} Y_n)$, where X_n, Y_n are finite-dimensional subspaces. Taking X_n+V instead of X_n and Y_n+W instead of Y_n we can assume that $V\subset X_n$ and $W\subset Y_n$ for $n=1, 2, \ldots$. Now let $M\in\mathscr{L}(S, V)$ and $N\in\mathscr{L}(Z, W)$ be fixed. Put for $n=1, 2, \ldots$, $M_n=M|_{X_n}$, $N_n=N|_{Y_n}$, $S_n=S\cap X_n$, and $Z_n=Z\cap Y_n$. By Lemma 2.3, α is a reasonable crossnorm on $X_n\otimes Y_n$ for $n=1, 2, \ldots$. Hence, by the first part of the proof applied to, M_n, N_n, S_n, and Z_n,

$$\lambda_{M_n\otimes N_n}(V\otimes_\alpha W, X_n\otimes_\alpha Y_n)\geqslant\lambda_{M_n}(V, X_n)\,\lambda_{N_n}(W, Y_n) \tag{2.7}$$

for $n=1, 2, \ldots$. By Lemma 2.2, $X\otimes_\alpha Y=\operatorname{cl}(\bigcup_{n=1}^{\infty}(X_n\otimes_\alpha Y_n))$. By the separability of X and Y, reasoning as in [19, Theorem 3.1.6, p. 85], we have

$$\lambda_M(V, X)=\lim_n \lambda_{M_n}(V, X_n), \qquad \lambda_N(W, Y)=\lim_n \lambda_{N_n}(W, Y_n)$$

and

$$\lambda_{M\otimes N}(V\otimes_\alpha W, X\otimes_\alpha Y)=\lim_n \lambda_{M_n\otimes N_n}(V\otimes_\alpha W, X_n\otimes_\alpha Y_n).$$

Hence, taking the limit over n on the both sides of (2.7) we get

$$\lambda_{M\otimes N}(V\otimes_\alpha W, X\otimes_\alpha Y)\geqslant\lambda_M(V, X)\,\lambda_N(W, Y),$$

which completes the proof.

THEOREM 2.6. *Let X, Y, S, Z, V, W, M, and N be as in Theorem 2.5. Assume that α is a reasonable, uniform crossnorm. Then*

$$\lambda_{M\otimes N}(V\otimes_\alpha W, X\otimes_\alpha Y)=\lambda_M(V, X)\,\lambda_N(W, Y).$$

Proof. Let $P_1 \in \mathscr{P}_M(X, V)$ and $P_2 \in \mathscr{P}_N(Y, W)$ be minimal extensions of M and N resp. Then $P_1 \otimes P_2 \in \mathscr{P}_{M \otimes N}(V \otimes_\alpha W, X \otimes_\alpha Y)$. Since α is uniform

$$\|P_1 \otimes P_2\|_\alpha \leqslant \|P_1\| \, \|P_2\| = \lambda_M(V, X) \, \lambda_N(W, Y).$$

The proof is complete.

By the induction argument one can easily deduce from Theorems 2.5 and 2.6 the following

THEOREM 2.7. *Let for $i = 1, ..., n$, X_i be a Banach space and let V_i be a finite dimensional subspace. Suppose that $V_i \subset S_i \subset X_i$ and let $M_i \in \mathscr{L}(S_i, V_i)$ be given. If α is a reasonable crossnorm on $\bigotimes_{i=1}^n X_i$ then*

$$\lambda_{\otimes_{i=1}^n M_i}\left(\bigotimes_{i=1}^n V_i, \bigotimes_{i=1}^n X_i\right) \geqslant \prod_{i=1}^n \lambda_{M_i}(V_i, X_i). \tag{2.8}$$

If α is a reasonable, uniform crossnorm then

$$\lambda_{\otimes_{i=1}^n M_i}\left(\bigotimes_{i=1}^n V_i, \bigotimes_{i=1}^n X_i\right) = \prod_{i=1}^n \lambda_{M_i}(V_i, X_i). \tag{2.9}$$

Remark 2.8. By [12, Theorem 3, p. 371] it is impossible to generalize Theorem 2.5 to the case of V being an arbitrary subspace of X and W being an arbitrary subspace of Y.

Remark 2.9. The constant $\lambda_{m \otimes N}(V \otimes_\alpha W, X \otimes_\alpha Y)$ does not depend on α for M, N, V, W, X and Y being fixed. Here α is a uniform reasonable crossnorm.

Remark 2.10. In [22, Corollary 14.1, p. 135] has been shown that for any pair of finite-dimensional Banach spaces V and W

$$\lambda(V \otimes_\lambda W) = \lambda(V) \, \lambda(W).$$

3

In this section we present some applications of Theorems 2.5–2.7. First we restrict ourselves to the case of minimal projections. By Theorems 1.5, 1.6, and 2.6 it is easy to prove

THEOREM 3.1. *Let S, T be compact, metrizable Hausdorff spaces. If V is a finite-dimensional subspace of $C(S)$ and W is a finite-dimensional subspace of $C(T)$ then*

$$\lambda(V \otimes_\lambda W, C(S) \otimes_\lambda C(T)) = \lambda(V, C(S)) \, \lambda(W, C(T)). \tag{3.1}$$

If S, T are σ-finite, separable measure spaces and V (W resp.) is a finite-dimensional subspace of $L_1(S)$, ($L_1(T)$ resp.) then

$$\lambda(V \otimes_\gamma W, L_1(S) \otimes_\gamma L_1(T)) = \lambda(V, L_1(S))\, \lambda(W, L_1(T)). \tag{3.2}$$

Now, let P_n denote the space of all polynomials of one real variable of degree $\leqslant n$ and $P_{n,m}$ the space of all polynomials of two variables of degree $\leqslant n$ with respect to the first variable and degree $\leqslant m$ with respect to the second variable. By the proof of Theorems 1.5 and 1.6 (see [13, pp. 9–11]) we have

$$P_n \otimes_\lambda P_m = P_{n,m} \tag{3.3}$$

and

$$P_n \otimes_\gamma P_m = P_{n,m}. \tag{3.4}$$

Here in (3.3) we consider P_n and P_m as subspaces of $C[-1, 1]$ and $P_{n,m}$ as a subspace of $C[-1, 1]^2$. In (3.4), P_n and P_m are subspaces of $L_1[-1, 1]$ and $P_{n,m}$ is a subspace of $L_1[-1, 1]^2$. Hence, by Theorem 3.1, if we know the projection constants $\lambda(P_n, C[-1, 1])$ or $\lambda(P_n, L_1[-1, 1])$ for the case of one variable, we know the relative projection constant of $P_{n,m}$ with respect to the supremum norm or to the L_1-norm. Also, by Theorem 2.6, if Q_1, Q_2 are minimal projections in the case of one variable, the tensor product of them is a minimal projection. Now, we present some examples when relative projection constants as well as formulas for minimal projections are known in the case of one variable in the L_1 or the supremum norms.

EXAMPLE 3.2. It is well known that

$$\lambda(P_1, C[-1, 1]) = 1.$$

Moreover, the interpolating projection with nodes in -1 and 1 is a minimal projection.

EXAMPLE 3.3. In [4] the minimal projection from $C[-1, 1]$ onto quadratics has been determined. In this case

$$\lambda(P_2, C[-1, 1]) = 1.2201730 \cdots$$

EXAMPLE 3.4. In [11] the minimal projection from $L_1[-1, 1]$ onto the lines P_1 has been found. In this case

$$\lambda(P_1, L_1[-1, 1]) = 1.22040 \cdots$$

EXAMPLE 3.5. In [9] the minimal projections from $L_1[-1, 1]$ onto P_n for $n=2, 3, 4, 5$ have been determined. The corresponding values of the relative projections constants are

$$\lambda(P_2, L_1[-1, 1]) = 1.36149\cdots$$

$$\lambda(P_3, L_1[-1, 1]) = 1.46184\cdots$$

$$\lambda(P_4, L_1[-1, 1]) = 1.54874\cdots$$

$$\lambda(P_5, L_1[-1, 1]) = 1.61031\cdots$$

EXAMPLE 3.6 [25]. Let n be an odd number. In this paper a minimal projection from $X_n = \text{Span}[t^n, t^2, t, 1]$ onto P_2 has been found in the case of the supremum norm on the interval $[-1, 1]$. By Theorem 3.1 and the previous considerations

$$\lambda(P_{2,2}, X_n \otimes_\lambda X_m) = \lambda(P_2, X_n)\, \lambda(P_2, X_m),$$

where in the space $X_n \otimes_\lambda X_m$ we consider the supremum norm on $[-1, 1]^2$.

EXAMPLE 3.7. In [24, Theorem 3] the following result has been shown,

$$\begin{aligned}
&\left(\frac{4(\ln n - \ln\ln n)}{\pi^2} + 1/3\right)\left(\frac{4(\ln m - \ln\ln m)}{\pi^2} + 1/3\right)\\
&\quad \leqslant \lambda(P_{n,m}, C[-1, 1]^2)\\
&\quad \leqslant \|T_n^1 \otimes_\lambda T_m^1\|\\
&\quad \leqslant \left(\frac{4\ln(2n+1)}{\pi^2} + 1\right)\left(\frac{4\ln(2m+1)}{\pi^2} + 1\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{4(\ln n - \ln\ln n)}{\pi^2} + 1/4\right)\left(\frac{4(\ln m - \ln\ln m)}{\pi^2} + 1/4\right)\\
&\quad \leqslant \lambda(P_{n,m}, L_1[-1, 1]^2)\\
&\quad \leqslant \|T_n^2 \otimes_\gamma T_m^2\|\\
&\quad \leqslant \left(\frac{4\ln(2n+3)}{\pi^2} + 1\right)\left(\frac{4\ln(2m+3)}{\pi^2} + 1\right),
\end{aligned}$$

where T_n^i denotes the nth partial sum operator of the Chebyshev expansion of the ith kind, $i=1, 2$. Theorem 3.1 permits us to reprove this result in a very simple manner. It is necessary to apply Theorems 1 and 2 from [24], where the necessary estimates for the case of one variable have been proved. Also, since

$$\lim_n \frac{\ln(2n+1)+\pi^2/4}{\ln n - \ln\ln n + \pi^2/12} = \lim_n \frac{\ln(2n+3)+\pi^2/4}{\ln n - \ln\ln n + \pi^2/16} = 1$$

by [24, Theorem 3], Theorem 4 from [24] is proved without applying [24, Lemmas 1 and 2].

Now we discuss the case $X=l_1^{(n)}$ and $Y=l_1^{(m)}$. Since by Theorem 1.6,

$$l_1^{(n)} \otimes_\gamma l_1^{(m)} = l_1^{(nm)},$$

if we know the formulas for minimal projections for some class of subspaces of $l_1^{(n)}$, then we know the formulas for minimal projections for tensor products of the spaces from this class with respect to the γ norm. The same remark applies to the case of subspaces of $l_\infty^{(n)}$ (here the λ norm should be used). Note that the formulas for minimal projections onto hyperplanes of $l_1^{(n)}$ have been found in [1]. See also [2, 3] for some formulas in the case of symmetric subspaces of $l_1^{(n)}$. In [1] the formulas for minimal projections onto hyperplanes of $l_\infty^{(n)}$ have been established. See also [18–20] where the case of subspaces of codimension two has been discussed. Also in [8] formulas for minimal projections onto some two-dimensional symmetric subspaces of $l_\infty^{(6)}$ have been presented.

EXAMPLE 3.8. Let Q_3 be a minimal projection from P_3 onto P_2 found in [25] (see Example 3.6). Put $V=P_2$, $S=P_3$, and $M=Q_3$. In [15] the constant $\lambda_M(V, P_4)$ has been calculated. Hence by Theorem 2.6 we have the formula for $\lambda_{M\otimes M}(P_{2,2}, P_{4,4})$.

EXAMPLE 3.9. In [10], it has been shown that if V is a two-dimensional, real normed space having unconditional basis v_1, v_2 and $M \in \mathscr{L}(V)$ is such that $Mv_i = d_i v_i$ then

$$\lambda_M(V) \leqslant (|d_1| + |d_2| + \sqrt{d_1^2 - |d_1 d_2| + d_2^2})/3. \tag{3.5}$$

Note that by [6],

$$\lambda_M(V) = \lambda_M(V, L_1[-1, 1]).$$

Also in [10, p. 174] the space V_M for which we have the equality in (3.5) has been described. Hence, by Theorem 2.6 for any M, N as above

$$\lambda_{M \otimes N}(V_M \otimes_\gamma V_N, L_1[-1,1]^2) \\ = \lambda_M(V_M, L_1[-1,1])\, \lambda_N(V_N, L_1[-1,1]).$$

Now we restrict ourselves to the case of L_p-spaces. We start with

DEFINITION 3.10 (see, e.g., [13, Definition 1.45, p. 27]). Let X, Y be Banach spaces. For $1 \leqslant p \leqslant \infty$ the p-nuclear norm of $z \in X \otimes Y$ is defined by

$$\alpha_p(z) = \inf \left\{ \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p} a_q(y_1, ..., y_n) : z = \sum_{i=1}^{n} x_i \otimes y_i \right\}. \tag{3.6}$$

Here q is so chosen that $1/p + 1/q = 1$ and

$$a_q(y_1, ..., y_n) = \sup \left\{ \left(\sum_{i=1}^{n} |f(y_i)|^q \right)^{1/q} : f \in S_{X^*} \right\}.$$

If $q = \infty$, then

$$a_q(y_1, ..., y_n) = \sup \left\{ \max_{1 \leqslant i \leqslant n} |f(y_i)| : f \in S_{X^*} \right\}.$$

By [13, Lemma 1.46, p. 27] the p-nuclear norm is a reasonable crossnorm. Observe that by [13, Lemma 1.44, p. 27] for any $B \in \mathscr{L}(Y)$

$$a_q(By_1, ..., By_n) \leqslant \|B\|\, a_q(y_1, ..., y_n).$$

Hence α_p is a uniform, reasonable crossnorm. By a result of [23] we have

$$L_p(S) \otimes_{\alpha_p} L_p(T) = L_p(S \times T), \tag{3.7}$$

where S and T are finite measure spaces. This enables us to apply Theorem 2.6 in the case of L_p-spaces. Until the end of this section S will stand for a finite separable measure space and let for every $n \in \mathbb{N}$, $(S)^n$ be a partition of S. Without loss, we can assume that each $Z \in (S)^n$ is a finite sum of elements from $(S)^{n+1}$. Let X_n be the space spanned by characteristic functions of the sets from $(S)^n$. Hence $X_n \subset X_{n+1}$. We can choose $(S)^n$ in such a way that $L_p(S) = \mathrm{cl}(\bigcup_{n=1}^{\infty} X_n)$ for $1 \leqslant p < \infty$. Note that, by Jensen's inequality for every $n \in \mathbb{N}$, a projection $P_n \in \mathscr{P}(L_p(S), X_n)$ defined by

$$P_n x = \sum_{Z \in (S)^n} \left(\int_Z x(s)\, d\mu(s)/\mu(Z) \right) \chi_Z \tag{3.8}$$

has norm one.

LEMMA 3.11. *If $f \in X_n$ and $g \in L_p(S)$ then*

$$\int_S f(t)\, g(t)\, d\mu(t) = \int f(t)(P_k g)(t)\, d\mu(t)$$

for any $k \geqslant n$.

Proof. Note that for $k \geqslant n$

$$\int_S f(t)(P_k g)(t)\, d\mu(t)$$

$$= \int_S \left\{ \sum_{Z \in (S)^k} \left[\int_Z g(s)\, d\mu(s)/\mu(Z) \right] (\chi_Z)(t) \right\} f(t)\, d\mu(t)$$

$$= \sum_{Z \in (S)^k} \left(\int_Z \left[\int_Z f(s)\, g(s)\, d\mu(s)/\mu(Z) \right] (\chi_Z)(t)\, d\mu(t) \right)$$

$$= \sum_{Z \in (S)^k} \int_Z g(s)\, f(s)\, d\mu(s) = \int_S f(t)\, g(t)\, d\mu(t),$$

as required.

LEMMA 3.12. *Let $f_1, ..., f_k$ be linearly independent, simple, measurable functions on S. Fix $1 < p < \infty$. Let $V = \bigcap_{i=1}^k \ker(f_i)$, where $\ker f_i$ denotes the kernel of f_i. Put*

$$V_n = V \cap X_n. \tag{3.9}$$

Then

$$\lambda(V, L_p(S)) = \lim_n \lambda(V_n, L_p(S)).$$

Proof. Let $P \in \mathscr{P}(L_p(S), V)$. Take $Q_n = P_n \circ P$. Since f_i are simple functions, modifying X_n, if necessary, we can assume that $f_i \in X_{n_o}$ for $i = 1, 2, ..., k$. By Lemma 3.11, for any $x \in L_p(S)$, $(P_n \circ P)\, x \in V_n$ for $n \geqslant n_o$. Since for any $x \in V_n$, $Q_n x = x$, $Q_n \in \mathscr{P}(L_p(S), V_n)$. Hence, since $\|P_n\| = 1$,

$$\limsup_n \lambda(V_n, L_p(S)) \leqslant \lambda(V, L_p(S)).$$

To prove the converse let $L_n \in \mathscr{P}(L_p(S), V_n)$ be a minimal projection and let (x_k) be a basis of $X = \bigcup_{n=1}^\infty X_n$. Since $1 < p < \infty$, by the diagonal argument and the Šmulian Theorem, we can assume that for fixed k, $L_n x_k$ converges weakly to the element which we denote by Px_k. Hence for any $x \in X$

$$\|Px\| = \left\| P\left(\sum_{i=1}^{l} a_i x_i\right)\right\| = \|\lim_n L_n x\|$$
$$\leqslant \liminf_n \|L_n x\| \leqslant \liminf_n \lambda(V_n, L_p(S)) \, \|x\|.$$

Consequently, by the density of X in $L_p(S)$, we can extend P onto all of $L_p(S)$. By the Mazur theorem, $Px \in V$ for any $x \in X$, and $Pv = v$ for any $v \in \bigcup_{n=1}^{\infty} V_n$. By Lemma 3.11, $\mathrm{cl}(\bigcup_{n=1}^{\infty} V_n) = V$. Hence $P \in \mathcal{P}(L_p(S), V)$ and consequently,

$$\lambda(V, L_p(S)) \leqslant \liminf_n \lambda(V_n, L_p(S)),$$

which completes the proof.

THEOREM 3.13. *Let* $f_1, ..., f_k$ ($g_1, ..., g_l$ *resp.*) *be a collection of linearly independent, simple measurable functions on* S (T *resp.*). *Fix* $1 < p < \infty$. *Put* $V = \bigcap_{i=1}^{k} \ker(f_i)$ *and* $W = \bigcap_{i=1}^{l} \ker(g_i)$. *Then*

$$\lambda(\mathrm{cl}(V \otimes W), L_p(S) \otimes_{\alpha_p} L_p(T)) = \lambda(V, L_p(S)) \, \lambda(W, L_p(T)).$$

Proof. For simplicity, let $U = L_p(S) \otimes_{\alpha_p} L_p(T)$ and $Z = \mathrm{cl}(V \otimes W)$, where the closure is taken with respect to the α_p-norm. Without loss, we also can assume that $S = T$. Let $Q_1 \in \mathcal{P}(L_p(S), V)$ and $Q_2 \in \mathcal{P}(L_p(S), W)$ be minimal projections. (By [14], minimal projections exist in our case.) Since $Q_1 \otimes_{\alpha_p} Q_2 \in \mathcal{P}(U, Z)$ and α_p is a uniform crossnorm,

$$\lambda(Z, U) \leqslant \lambda(V, L_p(S)) \, \lambda(W, L_p(S)).$$

To prove the converse, suppose that

$$\lambda(Z, U) < \lambda(V, L_p(S)) \, \lambda(W, L_p(S)). \tag{3.10}$$

Let $Q \in \mathcal{P}(U, Z)$ be a minimal projection. Without loss, we can assume that the spaces X_n are so chosen that f_j and $g_i \in X_{n_o}$ for $j = 1, ..., k$ and $i = 1, ..., l$. Put for $n \in N$

$$L_n = (P_n \otimes_{\alpha_p} P_n) \circ Q. \tag{3.11}$$

Note that, by Lemma 3.11, for any $x \in U$ and $n \geqslant n_o L_n x \in W_n$ (see (3.9)). Also for any $z \in V_n \otimes W_n$, $L_n z = z$. This shows that $L_n \in \mathcal{P}(U, V_n \otimes W_n)$. By (3.10), since $\|P_n \otimes_{\alpha_p} P_n\| \leqslant \|P_n\|^2 = 1$,

$$\lambda(V_n \otimes W_n, U) \leqslant \lambda(Z, U) < \lambda(V, L_p(S))\, \lambda(W, L_p(S)). \tag{3.12}$$

By Theorem 2.6 applied to the α_p-norm and Lemma 3.12,

$$\begin{aligned}\lim_n \lambda(V_n \otimes W_n, U) &= \lim_n (\lambda(V_n, L_p(S))\, \lambda(W_n, L_p(S)))\\ &= \lambda(V, L_p(S))\, \lambda(W, L_p(S)),\end{aligned}$$

a contradiction with (3.12). The proof is complete.

Since the space $L_p(S) \otimes_{\alpha_p} L_p(T)$ is linearly isometric to $L_p(S \times T)$, Theorem 3.13 permits us to calculate or estimate the relative projection constant for a class of subspaces of $L_p(S \times T)$ of infinite dimension and codimension provided we know the value or estimate for $\lambda(V, L_p(S))$ and $\lambda(W, L_p(T))$. Note that in [16] the relative projection constant onto any hyperplane of $L_p[0, 1]$ has been calculated. In fact, by a result of Rolewicz (see [21, Theorem II.7.5, p. 83; 26]), since $L_p[0, 1]$ is an almost isotropic space, the relative projection constant onto any hyperplane is the same and it is equal to

$$\max_{t \in [0, 1]} (t^{p-1} + (1-t)^{p-1})^{1/p}\, (t^{q-1} + (1-t)^{q-1})^{1/q}, \tag{3.13}$$

where q is so chosen that $1/p + 1/q = 1$. Also in [17] it has been shown that the number from (3.13) is a lower bound of the relative projection constant of any *rich* subspace of $L_p[0, 1]$. Note that by [17, Theorem 2], any subspace of finite codimension in $L_p[0, 1]$ is rich.

At the end of this paper we present a method of constructing various uniform crossnorms on $X \otimes Y$.

PROPOSITION 3.14. *Let $n \in \mathbb{N}$ and let $\|\cdot\|_n$ be a norm on $\mathbb{R}^n$ satisfying the order preserving condition, i.e., $\|(x_1, ..., x_n)\|_n \leqslant \|(y_1, ..., y_n)\|_n$ provided $|x_i| \leqslant |y_i|$ for $i = 1, ..., n$. If $\alpha_1, ..., \alpha_n$ are uniform crossnorms on $X \otimes Y$ then a function*

$$\alpha(z) = \|(\alpha_1(z), ..., \alpha_n(z))\|_n / \|(1, 1, ..., 1)\|_n$$

is a uniform crossnorm on $X \otimes Y$.

The proof of Proposition 3.14 is straightforward, so we omit it.

Remark 3.15. By Theorem 2.7, all the results from Section 3 concerning the tensor product of two Banach spaces hold true for the case of the tensor product of n Banach spaces $X_1, ..., X_n$.

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